Pole Placement Design

EE 432 Lecture Note

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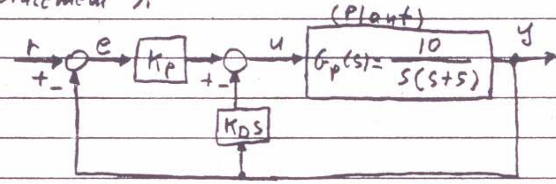
Pole Placement Design

Idea: Use compensator with "open" parameters to be adjusted such that the characteristic equation (of the closed-loop system) has a set of specified roots.
(Note that the roots (zeros) of the characteristic equation are poles of the (closed-loop) transfer function; hence, the name "pole placement").

Example 1:



Obtain the values of gains K_p and K_D such that the closed-loop system has a characteristic equation of the form $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ with $\zeta = 0.8$ and $\omega_n = 5$, i.e., the characteristic equation $s^2 + 8s + 25 = 0$ [roots: $s_1, s_2 = -4 \pm j3$];

From diagram: characteristic equation:

$$1 + K_p \frac{G_p(s)}{1 + K_D s G_p(s)} = 1 + \frac{\frac{10K_p}{s(s+5)}}{1 + \frac{10K_D s}{s(s+5)}} = 1 + \frac{10K_p}{s^2 + (5 + 10K_D)s} = 0, \text{ or}$$

$s^2 + (5 + 10K_D) + 10K_p = 0$. We want $s^2 + 8s + 25 = 0$

Equating terms with equal power of s: $\left.\begin{aligned} s^0 &: 10K_p = 25 \\ s^1 &: 5 + 10K_D = 8 \end{aligned}\right\}$

$\therefore$ $K_p = 2.5$, $K_D = 0.3$

Example 2: Express the system of Example 1 in state representation (use control-canonical form) and obtain K_p and K_D for the same specifications [char. eq.: $s^2 + 8s + 25 = 0$].

Plant: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$; $y = 10x_1 = [10, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Feedback: $u = K_p(r - y) - K_D \dot{y} = K_p r - 10K_p x_1 - 10K_D x_2 = K_p r - [10K_p, 10K_D] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(since $y = 10x_1$, $\dot{y} = 10\dot{x}_1 = 10x_2$)

Thus, the closed-loop state equation becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(K_p r - [10K_p, 10K_D] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} K_p r - \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} [10K_p, 10K_D]}_{\begin{bmatrix} 0 & 0 \\ 10K_p & 10K_D \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 10K_p & 10K_D \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_p \end{bmatrix} r$$

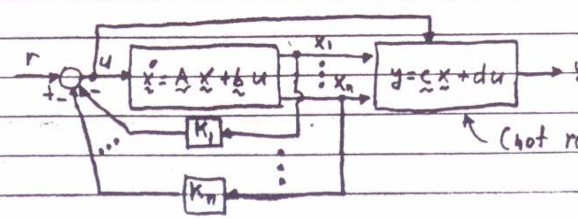
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10K_p & (-5 - 10K_D) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_p \end{bmatrix} r = \underset{\sim}{A}_c \underset{\sim}{x} + \underset{\sim}{b}_c r$$

where A_c and b_c are the A- and b-matrices respectively of the closed-loop feedback system, whose char. eq. becomes

$$|s\underset{\sim}{I} - \underset{\sim}{A}_c| = \left| s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -10K_p & (-5 - 10K_D) \end{bmatrix} \right| = \begin{vmatrix} s & -1 \\ 10K_p & (s + 10K_D + 5) \end{vmatrix} = 0$$

or: $s^2 + (10K_D + 5)s + 10K_p = 0$ (as in Example 1), which, when compared with the specified char. eq. $s^2 + 8s + 25 = 0$ yields $K_p = 2.5$ and $K_D = 0.3$ (as before).

Full-State (FS) Feedback for Single-Input (SI) Plants



(not relevant here)

Consider full-state feedback, given by

$$u = r - \sum_{i=1}^{n} K_i x_i = r - [K_1 K_2 \cdots K_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = r - \underset{\sim}{K}\,\underset{\sim}{x} \qquad \text{(Control)}$$

where $\underset{\sim}{K} = [K_1 K_2 \cdots K_n]$

This control is applied to the (open-loop) plant:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{b}\,u \;;\quad y = \underset{\sim}{c}\,\underset{\sim}{x} \qquad \text{(Plant)}$$

to yield the state eqs. of the (closed-loop) feedback system:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{b}(r - \underset{\sim}{K}\,\underset{\sim}{x}) = (\underset{\sim}{A} - \underset{\sim}{b}\,\underset{\sim}{K})\underset{\sim}{x} + \underset{\sim}{b}\,r \triangleq \underset{\sim}{A}_c\,\underset{\sim}{x} + \underset{\sim}{b}\,r \qquad \left(\begin{matrix}\text{Feedback}\\ \text{System}\end{matrix}\right)$$

where $\underset{\sim}{A}_c = \underset{\sim}{A} - \underset{\sim}{b}\,\underset{\sim}{K}$ = A-matrix of the feedback system.

Advantage of FS-feedback: Unique solution (n gains K_i to obtain n specified roots s_i $(i=1,2,\ldots,n)$)

Disadvantage: All state variables must be available for feedback - usually not possible or practical (Observer design, to be discussed later, may overcome the problem).

Pole-Placement for FS Feedback:

Let $s_1, s_2, \ldots, s_n$ be the roots which the characteristic equation is to have. Thus, the desired characteristic equation of the closed-loop system,

$$\alpha_c(s) = |s\underset{\sim}{I} - \underset{\sim}{A}_c| = |s\underset{\sim}{I} - \underset{\sim}{A} + \underset{\sim}{b}\,\underset{\sim}{K}| = 0$$

can be specified in terms of

$$\alpha_c(s) = (s-s_1)\cdots(s-s_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$$

Example 3: Consider the same system of Examples 1 and 2, in the same (control-canonical) form of Example 2, but with FS feedback of the form described above. Determine K_1 and K_2 to satisfy the same specifications as before, i.e., $\alpha_c(s) = s^2+8s+25=0$.

We have $\underset{\sim}{A}_c = \underset{\sim}{A} - \underset{\sim}{b}\,\underset{\sim}{K} = \begin{bmatrix}0 & 1\\0 & -5\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[K_1 K_2]$

$$= \begin{bmatrix}0 & 1\\0 & -5\end{bmatrix} - \begin{bmatrix}0 & 0\\K_1 & K_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-K_1 & (-K_2-5)\end{bmatrix}$$

$$\therefore \alpha_c(s) = |s\underset{\sim}{I} - \underset{\sim}{A}_c| = \begin{vmatrix}s & -1\\K_1 & (s+K_2+5)\end{vmatrix} = s^2 + (K_2+5)s + K_1 = 0$$

We want: $\alpha_c(s) = s^2 + 8s + 25 = 0$

$$\therefore \left.\begin{matrix}K_1 = 25\\ K_2+5 = 8 \;\; \therefore K_2 = 3\end{matrix}\right\} \text{ i.e., } \underset{\sim}{K} = [25, 3]$$

> Note that in Example 2, we had $u = r - K_p y - K_D \dot{y}$ with $y = 10x_1$ and $\dot{y} = 10\dot{x}_1 = 10x_2$; thus, $u = r - 10K_p x_1 - 10K_D x_2$. Here (in Example 3), we have (for the same state-variable form) $u = r - K_1 x_1 - K_2 x_2$. Therefore, $K_1 = 10K_p$, $K_2 = 10K_D$

We see that the FS feedback design boils down to the problem of solving for $\underset{\sim}{K} = [K_1 \ldots K_n]$ for given $\underset{\sim}{A}$, $\underset{\sim}{b}$ and (specified) root-coefficients $\alpha_0, \ldots, \alpha_{n-1}$, such that

$$\boxed{\alpha_c(s) = |s\underset{\sim}{I} - \underset{\sim}{A} + \underset{\sim}{b}\,\underset{\sim}{K}| = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0}$$

FS Feedback for Plant in Control-Canonical Form:

$$u \longrightarrow \boxed{G_p(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}} \longrightarrow y$$

In the control-canonical form, the state eq. for the plant is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{\underset{\sim}{A}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\underset{\sim}{b}} u$$

(Note that for FS feedback, the output equation, $y = \underset{\sim}{c}\,\underset{\sim}{x} + du$, is not relevant)

$$\underset{\sim}{A}_c = \underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{K} = \underset{\sim}{A} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [K_1 K_2 \cdots K_n] = \underset{\sim}{A} - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ K_1 & K_2 & K_3 & \cdots & K_{n-1} & K_n \end{bmatrix}$$

or:

$$\underset{\sim}{A}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ (-a_0-K_1) & (-a_1-K_2) & (-a_2-K_3) & \cdots & (-a_{n-2}-K_{n-1}) & (-a_{n-1}-K_n) \end{bmatrix}$$

Thus, the characteristic equation becomes:

$$\alpha_c(s) = |s\underset{\sim}{I} - \underset{\sim}{A}_c| = \begin{vmatrix} s & (-1) & 0 & \cdots & 0 & 0 \\ 0 & s & (-1) & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & s & (-1) \\ (a_0+K_1) & (a_1+K_2) & (a_2+K_3) & \cdots & (a_{n-2}+K_{n-1}) & (s+a_{n-1}+K_n) \end{vmatrix} = 0$$

It can be shown that this yields

$$\alpha_c(s) = s^n + (a_{n-1}+K_n)s^{n-1} + \cdots + (a_1+K_2)s + (a_0+K_1) = 0$$

From the specified roots, the desired char. eq. should be

$$\alpha_c(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$$

Equating terms in equal power of s yields,

$$\begin{aligned} a_0 + K_1 = \alpha_0 &\Rightarrow K_1 = \alpha_0 - a_0 \\ a_1 + K_2 = \alpha_1 &\Rightarrow K_2 = \alpha_1 - a_1 \\ &\ \ \vdots \\ a_{n-1} + K_n = \alpha_{n-1} &\Rightarrow K_n = \alpha_{n-1} - a_{n-1} \end{aligned}$$

(for plant in control-canonical form)

or,

$$\underset{\sim}{K}^T = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} - \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \underset{\sim}{\alpha} - \underset{\sim}{a}$$

Example 4: Repeat Example 3 by using the above equations. Since the plant is in control-canonical form, the above equations are valid. From the $\underset{\sim}{A}$-matrix of the plant, $\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$, we have $\underset{\sim}{a} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. For the specified char. eq., $\alpha_c(s) = s^2 + \alpha_1 s + \alpha_0 = s^2 + 8s + 25$, we have $\underset{\sim}{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 25 \\ 8 \end{bmatrix}$.

$$\therefore \underset{\sim}{K}^T = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 25 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 25 \\ 3 \end{bmatrix} \quad \text{(as before)}$$

FS Feedback Design for Plant in General Form

Again, the problem is:

For given A, b and specified coefficients $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$, find $K = [K_1, K_2, \dots, K_n]$ such that

$$\alpha_c(\alpha) = |sI - A + bK| = s^2 + \alpha_{n-1}s^{n-1} + \dots + \alpha_1 s + \alpha_0 = 0$$

We have seen that for the control-canonical form, the solution is rather easy to use. However, this may not be true in general. Two possibility exist: Either

- ~~Use similarity transform to transform a given form to the control-canonical form, solve for K of that form and then transform back to original form to find the K to be used.~~

or

- Transform the formulas for the control-canonical form into obtain formulas valid for any state-variable form.

The latter approach is preferable though the general formulas appear rather "long," they are easily mechanized by computer. One such result is Ackermann's formula [p.413, Phillips and Harbor, eq. (10-23)]:

$$\boxed{K = [0\,0 \dots 0\,1]\,Q^{-1}\alpha_c(A)}$$, where

$$Q = [b,\ Ab,\ \dots,\ A^{n-2}b,\ A^{n-1}b] = \text{"controllability matrix"}$$

and[*]

$$\alpha_c(A) = A^n + \alpha_{n-1}A^{n-1} + \dots + \alpha_1 A + \alpha_0 I$$

i.e., $\alpha_c(A)$ is the characteristic polynominal with s replaced by A.

The "controllability matrix" Q plays another important role to be discussed in the context of "controllability" of a plant.

Example 5: In Example 4, we had $A = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore Q = [b,\ Ab] = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \quad \therefore Q^{-1} = \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

The char. eq. to be satisfied was $\alpha_c(s) = s^2 + 8s + 25 = 0$

$$\therefore \alpha_c(A) = A^2 + 8A + 25I = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} + 8\begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} + 25\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -5 \\ 0 & 25 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ 0 & -40 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 25 & 3 \\ 0 & 10 \end{bmatrix}$$

$$\therefore K = [0\ 1]\,Q^{-1}\alpha_c(A) = \underbrace{[0\ 1]\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} 25 & 3 \\ 0 & 10 \end{bmatrix}$$

$$= [1\ 0]\begin{bmatrix} 25 & 3 \\ 0 & 10 \end{bmatrix} = [25,\ 3]$$

(as before)

[*] Note of interest: Cayley Hamilton Theorem: "Every square matrix satisfies its own characteristic equation", i.e., $|sI - A| = \sum_{i=0}^{n} \alpha_i s^i = 0$ implies $\sum_{i=0}^{n} \alpha_i A^i = 0$ with $A^0 = I$.